

Exercise 1-3 Answer

2.

证明

$D(x)$ 在对任意 $x_0 \in \mathbb{R}$, $D(x_0)$ 上有定义。

对每一个 n , 在区间 $(x_0, x_0 + \frac{1}{n})$ 上可取一个 $x_n \in \mathbb{Q}$ ($\mathbb{Q}$ 稠密), 有

$$x_0 < x_n < x_0 + \frac{1}{n}.$$

$$\lim_{n \rightarrow \infty} x_n = x_0 = \lim_{n \rightarrow \infty} \left(x_0 + \frac{1}{n}\right).$$

故

$$\lim_{n \rightarrow \infty} x_n = x_0, \quad \lim_{n \rightarrow \infty} D(x_n) = 1.$$

同理, 在区间 $(x_0, x_0 + \frac{1}{n})$ 上可取一个 $y_n \in \mathbb{R} \setminus \mathbb{Q}$ ($\mathbb{R}$ 稠密), 有

$$\lim_{n \rightarrow \infty} y_n = x_0, \quad \lim_{n \rightarrow \infty} D(y_n) = 0.$$

故 $D(x)$ 的两个子列 $\{D(x_n)\}, \{D(y_n)\}$ 在 x_0 点收敛于不同值。

由海涅定理, 对任意 $x_0 \in \mathbb{R}$, $D(x)$ 极限不存在。

5.

(1)

$$\lim_{x \rightarrow +\infty} \sin \sqrt{x}.$$

证明

假设 $\lim_{x \rightarrow +\infty} \sin \sqrt{x} = L \in [-1, 1]$ 。由极限定义, $\forall \varepsilon > 0$, $\exists X$, 当 $x > X$ 时,

$$|\sin \sqrt{x} - L| < \varepsilon,$$

即

$$L - \varepsilon < \sin \sqrt{x} < L + \varepsilon.$$

取 $x_1 = (2k\pi)^2$, $k \in \mathbb{Z}$, $x_1 > X$, 则

$$L - \varepsilon < \sin(2k\pi) = 0 < L + \varepsilon,$$

$$-\varepsilon < L < \varepsilon.$$

由 ε 的任意性, $L = 0$ 。

取 $x_2 = (2t\pi + \frac{\pi}{2})^2$, $t \in \mathbb{Z}$, $x_2 > X$, 则

$$L - \varepsilon < \sin\left(2t\pi + \frac{\pi}{2}\right) = 1 < L + \varepsilon,$$

$$1 - \varepsilon < L < 1 + \varepsilon.$$

由 ε 的任意性, $L = 1$, 矛盾。原极限不存在。

(2)

$$\lim_{x \rightarrow 0} \sin\left(\frac{1}{x^2}\right).$$

证明

假设 $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x^2}\right) = L$ 。由极限定义, $\forall \varepsilon > 0$, $\exists \delta$, 当 $0 < |x| < \delta$ 时,

$$\left| \sin\left(\frac{1}{x^2}\right) - L \right| < \varepsilon.$$

取 $x_1 = \frac{1}{\sqrt{2k\pi}} < \delta$, $k \in \mathbb{Z}$, 则

$$|\sin(2k\pi) - L| < \varepsilon \Rightarrow |L| < \varepsilon.$$

取 $x_2 = \frac{1}{\sqrt{2t\pi + \frac{\pi}{2}}} < \delta$, $t \in \mathbb{Z}$, 则

$$\left| \sin\left(2t\pi + \frac{\pi}{2}\right) - L \right| < \varepsilon \Rightarrow |1 - L| < \varepsilon.$$

由 ε 的任意性, $|L| < \varepsilon$ 与 $|1 - L| < \varepsilon$ 不可能同时成立, 矛盾。原极限不存在。

(4)

$$\lim_{x \rightarrow 0} e^{\frac{1}{x}}.$$

考虑 $\lim_{x \rightarrow 0^+} e^{\frac{1}{x}}$, 由

$$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty,$$

得

$$\lim_{x \rightarrow 0^+} e^{\frac{1}{x}} = +\infty.$$

考虑 $\lim_{x \rightarrow 0^-} e^{\frac{1}{x}}$, 由

$$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty,$$

得

$$\lim_{x \rightarrow 0^-} e^{\frac{1}{x}} = 0.$$

右侧极限不存在, 原极限不存在。

证明

因为

$$\lim_{n \rightarrow \infty} \frac{n^k}{a^n} = 0, \quad a > 1,$$

所以

$$\lim_{n \rightarrow \infty} \frac{n^k}{a^{n-1}} = 0.$$

而 $[x] \leq x < [x] + 1$ 。不妨考虑 $x > 0$, 则

$$0 < \frac{x^k}{a^x} < \frac{([x] + 1)^k}{a^{[x]}}.$$

而

$$\lim_{x \rightarrow \infty} \frac{([x] + 1)^k}{a^{[x]}} = 0,$$

由夹逼定理,

$$\lim_{x \rightarrow \infty} \frac{x^k}{a^x} = 0.$$

8.

(2)

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1+x)^5 - (1+5x)}{x^2 + x^5} \\ &= \lim_{x \rightarrow 0} \frac{x^5 + 5x^4 + 10x^3 + 10x^2}{x^2 + x^5} \\ &= \lim_{x \rightarrow 0} \frac{x^3 + 5x^2 + 10x + 10}{1 + x^3} = 10. \end{aligned}$$

(13)

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1+mx)^n - (1+nx)^m}{x^2}, \quad (m, n \in \mathbb{N}^*). \\ &= \lim_{x \rightarrow 0} \frac{\sum_{k=0}^n C_n^k m^k x^k - \sum_{t=0}^m C_m^t n^t x^t}{x^2}. \end{aligned}$$

由于高于 2 次的项都会化为 0, 当 $n, m \geq 2$, (验证 $m = 1$ 或 $n = 1$)。

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{[C_n^2 m^2 x^2 + C_n^1 m x + C_n^0 m] - [C_m^2 n^2 x^2 + C_m^1 n x + C_m^0 n]}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\left[\frac{n(n-1)}{2} m^2 - \frac{m(m-1)}{2} n^2 \right] x^2 + m - n}{x^2} \\ &= \begin{cases} \frac{n(n-1)}{2} m^2 - \frac{m(m-1)}{2} n^2 m = n & m = n \\ \text{不存在} & m \neq n \end{cases}. \end{aligned}$$

(15)

$$\lim_{x \rightarrow \infty} \frac{(x+1)(x^2+1) \cdots (x^n+1)}{[(nx)^n+1]^{\frac{n+1}{2}}}.$$

上下同除 $x^{\frac{n(n+1)}{2}}$:

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{\frac{\prod_{k=1}^n (x^k+1)}{x^{\frac{n(n+1)}{2}}}}{\frac{[(nx)^n+1]^{\frac{n+1}{2}}}{x^{\frac{n(n+1)}{2}}}}. \\ &= \lim_{x \rightarrow \infty} \frac{\prod_{k=1}^n \left(\frac{x^k+1}{x^k}\right)}{\left(n^n + \frac{1}{x^n}\right)^{\frac{n+1}{2}}}. \\ &= \frac{\lim_{x \rightarrow \infty} \prod_{k=1}^n \left(1 + \frac{1}{x^k}\right)}{\lim_{x \rightarrow \infty} \left(n^n + \frac{1}{x^n}\right)^{\frac{n+1}{2}}}. \\ &= \frac{1}{n^{\frac{n(n+1)}{2}}} = n^{-\frac{n(n+1)}{2}}. \end{aligned}$$

(8)

$$\lim_{x \rightarrow 1} \frac{x^{n+1} - (n+1)x + n}{(x-1)^2}.$$

由洛必达法则,

$$= \lim_{x \rightarrow 1} \frac{(n+1)x^n - (n+1)}{2x - 2}.$$

再由洛必达法则,

$$= \lim_{x \rightarrow 1} \frac{n(n+1)x^{n-1}}{2} = \frac{n(n+1)}{2}.$$

(9)

$$\begin{aligned} &\lim_{x \rightarrow +\infty} \frac{\sqrt{x + \sqrt{x + \sqrt{x}}}}{\sqrt{x + \sqrt{x}}} \\ &= \lim_{x \rightarrow +\infty} \sqrt{\frac{1}{1 + \frac{1}{\sqrt{x}}} + 1} = \sqrt{2}. \end{aligned}$$

(10)

$$\begin{aligned} &\lim_{x \rightarrow 4} \frac{\sqrt{1+2x} - 3}{\sqrt{x} - 2} \\ &= \lim_{x \rightarrow 4} \frac{(\sqrt{1+2x} - 3)(\sqrt{x} + 2)}{x - 4} \end{aligned}$$

$$\begin{aligned} &= \lim_{x \rightarrow 4} \frac{1 + 2x - 9}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{1 + 2x} + 3} \\ &= \lim_{x \rightarrow 4} \frac{2(\sqrt{x} + 2)}{\sqrt{1 + 2x} + 3} = \frac{4}{3}. \end{aligned}$$

9.

(1)

$$\lim_{x \rightarrow \pi} \frac{\sin(mx)}{\sin(nx)}, \quad (m, n \in \mathbb{Z}, n \neq 0).$$

$$\text{令 } t = x - \pi,$$

$$\begin{aligned} &= \lim_{t \rightarrow 0} \frac{\sin(m(t + \pi))}{\sin(n(t + \pi))} \\ &= \lim_{t \rightarrow 0} \frac{\sin(mt)}{\sin(nt)} (-1)^{m-n} \\ &= \lim_{t \rightarrow 0} \frac{\frac{\sin(mt)}{mt}}{\frac{\sin(nt)}{nt}} \cdot \frac{m}{n} (-1)^{m-n} \\ &= (-1)^{m-n} \frac{m}{n}. \end{aligned}$$

(3)

$$\lim_{x \rightarrow 1} (1 - x) \tan\left(\frac{\pi}{2}x\right).$$

$$\text{令 } t = 1 - x,$$

$$\begin{aligned} &= \lim_{t \rightarrow 0} t \tan\left(\frac{\pi}{2}(1 - t)\right) \\ &= \lim_{t \rightarrow 0} t \cot\left(\frac{\pi}{2}t\right) \\ &= \lim_{t \rightarrow 0} \frac{t}{\sin\left(\frac{\pi}{2}t\right)} \cdot \cos\left(\frac{\pi}{2}t\right) \\ &= 1. \end{aligned}$$

(5)

$$\begin{aligned} &\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos(x^2)}}{1 - \cos x} \\ &= \lim_{x \rightarrow 0} \frac{\sqrt{\frac{1}{2}x^4}}{\frac{1}{2}x^2} \\ &= \sqrt{2}. \end{aligned}$$

(8)

$$\begin{aligned} &\lim_{x \rightarrow \infty} \left(\frac{x+a}{x-a}\right)^x \\ &= \lim_{x \rightarrow \infty} \left[1 + \frac{2a}{x-a}\right]^x \end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{\frac{x}{2a} - \frac{1}{2}} \right)^{\frac{x}{2a} - \frac{1}{2}} \cdot \left(1 + \frac{1}{\frac{x}{2a} - \frac{1}{2}} \right)^{\frac{1}{2}} \right]^{2a} \\
&= e^{2a}.
\end{aligned}$$

10.

$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) = 0.$$

求 a, b 。

$$\begin{aligned}
&\lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) \\
&= \lim_{x \rightarrow \infty} \left(x - \frac{x}{x + 1} - ax - b \right) \\
&= \lim_{x \rightarrow \infty} \left[(1 - a)x - \frac{x}{x + 1} - b \right].
\end{aligned}$$

假设 $a \neq 1$,

$$\lim_{x \rightarrow \infty} [(1 - a)x - b] = \infty, \quad \lim_{x \rightarrow \infty} \frac{x}{x + 1} = 1.$$

不可能原极限为 0。

故 $a = 1$, 进而

$$\begin{aligned}
b &= - \lim_{x \rightarrow \infty} \frac{x}{x + 1} = -1. \\
&\Rightarrow a = 1, \quad b = -1.
\end{aligned}$$

11.

(1)

$$\begin{aligned}
&\lim_{x \rightarrow -\infty} \left(\sqrt{x^2 - x + 1} - a_1 x - b_1 \right) = 0. \\
&= \lim_{x \rightarrow -\infty} \left[-x \sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} - a_1 x - b_1 \right] \\
&= \lim_{x \rightarrow -\infty} (-x - a_1 x - b_1).
\end{aligned}$$

由原极限存在, $a_1 = -1$ 。

$$\begin{aligned}
&\Rightarrow \lim_{x \rightarrow -\infty} \left(\sqrt{x^2 - x + 1} + x - b_1 \right) = 0. \\
b_1 &= \lim_{x \rightarrow -\infty} \left(\sqrt{x^2 - x + 1} + x \right)
\end{aligned}$$

$$\begin{aligned}
&= \lim_{x \rightarrow -\infty} \frac{(x^2 - x + 1) - x^2}{\sqrt{x^2 - x + 1} - x} \\
&= \lim_{x \rightarrow -\infty} \frac{-x + 1}{\sqrt{x^2 - x + 1} - x} \\
&= \lim_{x \rightarrow -\infty} \frac{-1 + \frac{1}{x}}{-\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} - 1} \\
&= \frac{1}{2}.
\end{aligned}$$

综上, $a_1 = -1$, $b_1 = \frac{1}{2}$ 。

(2)

$$\begin{aligned}
&\lim_{x \rightarrow +\infty} (\sqrt{x^2 - x + 1} - a_2 x - b_2) = 0. \\
&= \lim_{x \rightarrow +\infty} x \left(\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} - a_2 - \frac{b_2}{x} \right).
\end{aligned}$$

原极限存在且为 0,

$$\Rightarrow \lim_{x \rightarrow +\infty} (1 - a_2) = 0 \Rightarrow a_2 = 1.$$

而

$$\begin{aligned}
&\lim_{x \rightarrow +\infty} (\sqrt{x^2 - x + 1} - x - b_2) \\
&= \lim_{x \rightarrow +\infty} \frac{(x^2 - x + 1) - (x + b_2)^2}{\sqrt{x^2 - x + 1} + x + b_2} \\
&= \lim_{x \rightarrow +\infty} \frac{(-1 - 2b_2)x + 1 - b_2^2}{\sqrt{x^2 - x + 1} + x + b_2} \\
&= \lim_{x \rightarrow +\infty} \frac{(-1 - 2b_2) + \frac{1 - b_2^2}{x}}{\sqrt{1 - \frac{1}{x} + \frac{1}{x^2}} + 1 + \frac{b_2}{x}} \\
&= \frac{-1 - 2b_2}{2} = 0. \\
&\Rightarrow a_2 = 1, \quad b_2 = -\frac{1}{2}.
\end{aligned}$$

12.

(1) 证明

要证 $\cos x = 1 - \frac{1}{2}x^2 + o(x^2)$ ($x \rightarrow 0$),

$$\Leftrightarrow \cos x - 1 + \frac{1}{2}x^2 = o(x^2), \quad (x \rightarrow 0).$$

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^2} \\
&= \lim_{x \rightarrow 0} \left(\frac{-\frac{1}{2}(1 - \cos x)}{x^2} + \frac{1}{2} \right) \\
&= -\frac{1}{2} + \frac{1}{2} = 0.
\end{aligned}$$

(2)

$$\begin{aligned}
& \lim_{x \rightarrow 0^+} \frac{\sqrt{x + \sqrt{x + \sqrt{x}}}}{x^{\frac{1}{8}}} \\
&= \lim_{x \rightarrow 0^+} \sqrt{x^{\frac{3}{4}} + \sqrt{x^{\frac{1}{2}} + 1}} = 1.
\end{aligned}$$

(3)

$$\lim_{x \rightarrow \infty} \frac{1 - \cos\left(\frac{1}{x}\right)}{\frac{1}{2x^2}}.$$

$$\text{令 } \frac{1}{x} = t,$$

$$\begin{aligned}
&= \lim_{t \rightarrow 0^+} \frac{1 - \cos t}{\frac{1}{2}t^2} \\
&= \lim_{t \rightarrow 0^+} \frac{2 \sin^2\left(\frac{t}{2}\right)}{2\left(\frac{t}{2}\right)^2} = 1.
\end{aligned}$$

(4)

$$o(x^n) \cdot o(x^m) = o(x^{m+n}), \quad (x \rightarrow 0).$$

设 $f(x) = o(x^n)$, $\lim_{x \rightarrow 0} \frac{f(x)}{x^n} = 0$; $g(x) = o(x^m)$, $\lim_{x \rightarrow 0} \frac{g(x)}{x^m} = 0$ 。则

$$\lim_{x \rightarrow 0} \frac{f(x) \cdot g(x)}{x^n \cdot x^m} = \lim_{x \rightarrow 0} \frac{f(x) \cdot g(x)}{x^{m+n}} = 0.$$

即

$$\begin{aligned}
& f(x) \cdot g(x) = o(x^{m+n}) \\
& \Leftrightarrow o(x^m) \cdot o(x^n) = o(x^{m+n}).
\end{aligned}$$

(5)

设 $f(x) = o(x^n)$, $\lim_{x \rightarrow 0} \frac{f(x)}{x^n} = 0$; $g(x) = o(x^m)$, $\lim_{x \rightarrow 0} \frac{g(x)}{x^m} = 0$, 且 $n > m > 0$ 。则

$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{g(x)}{x^n} = \lim_{x \rightarrow 0} \frac{g(x)}{x^m} \cdot \frac{1}{x^{n-m}} = 0. \\
& \lim_{x \rightarrow 0} \frac{f(x) + g(x)}{x^m} = \lim_{x \rightarrow 0} \frac{f(x)}{x^m} + \lim_{x \rightarrow 0} \frac{g(x)}{x^m} = 0.
\end{aligned}$$

即 $o(x^n) + o(x^m) = o(x^m)$ 。

13.

证明

“ $\Rightarrow$ ”: $\alpha(x) \sim \beta(x) \ (x \rightarrow x_0)$ 。由定义且 $x \in \dot{U}(x_0)$, $\beta(x) \neq 0$,

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{\alpha(x)}{\beta(x)} &= 1 \\ \Leftrightarrow \lim_{x \rightarrow x_0} \left[\frac{\alpha(x)}{\beta(x)} - 1 \right] &= 0 \\ \Leftrightarrow \lim_{x \rightarrow x_0} \frac{\alpha(x) - \beta(x)}{\beta(x)} &= 0 \\ \Leftrightarrow \alpha(x) - \beta(x) &= o(\beta(x)). \end{aligned}$$

“ $\leq$ ”: 由 $\alpha(x) = \beta(x) + o(\beta(x)) \ (x \rightarrow x_0)$,

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow x_0} \frac{\alpha(x) - \beta(x)}{\alpha(x)} &= 0 \\ \Rightarrow \lim_{x \rightarrow x_0} \left(\frac{\alpha(x)}{\beta(x)} - 1 \right) &= 0 \\ \Rightarrow \lim_{x \rightarrow x_0} \frac{\alpha(x)}{\beta(x)} &= 1. \end{aligned}$$

即 $\alpha(x) \sim \beta(x) \ (x \rightarrow x_0)$ 。

16.

(1)

$$\begin{aligned} &\lim_{x \rightarrow 0} \frac{\sqrt{1 + \tan^2 x} - 1}{x \sin x} \\ &= \lim_{x \rightarrow 0} \frac{(1 + \tan^2 x) - 1}{x \sin x (\sqrt{1 + \tan^2 x} + 1)} \\ &= \lim_{x \rightarrow 0} \frac{x^2}{x^2 (\sqrt{1 + \tan^2 x} + 1)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{1 + \tan^2 x} + 1} = \frac{1}{2}. \end{aligned}$$

(2)

$$\begin{aligned} &\lim_{x \rightarrow 0} \frac{\sin(\tan^2 x)}{\sqrt[n]{1 + x^2} - 1} \\ &= \lim_{x \rightarrow 0} \frac{x^2}{\frac{1}{n} x^2} = n. \end{aligned}$$

(3)

$$\begin{aligned} & \lim_{x \rightarrow 0^+} \frac{(1 - \sqrt{\cos x}) \tan x}{(1 - \cos x)^{\frac{3}{2}}} \\ &= \lim_{x \rightarrow 0^+} \frac{(1 - \cos x) \tan x}{(1 + \sqrt{\cos x})(1 - \cos x)^{\frac{3}{2}}} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{2}x^2 \cdot x}{(1 + \sqrt{\cos x})\left(\frac{1}{2}x^2\right)^{\frac{3}{2}}} \\ &= \lim_{x \rightarrow 0^+} \frac{\frac{1}{2}}{(1 + \sqrt{\cos x})\left(\frac{1}{2}\right)^{\frac{3}{2}}} = \frac{\sqrt{2}}{2}. \end{aligned}$$

Exercise 1-4 Answer

3.

(3)

$$f(x) = e^{\frac{1}{x}}.$$

第二类间断点 $x = 0$ (无穷间断点)。

(4)

$$f(x) = \frac{1}{x} \sin\left(\frac{1}{x}\right).$$

第二类间断点 $x = 0$ 。

(7)

$$f(x) = \arctan\left(\frac{1}{x}\right).$$

跳跃间断点 $x = 0$ 。

(10)

$$\begin{aligned} f(x) &= \lim_{n \rightarrow \infty} \frac{x^n}{1 + x^n}, \quad (x \geq 0) \\ &= \begin{cases} 0, & 0 \leq x < 1 \\ \frac{1}{2}, & x = 1 \\ 1, & x > 1 \end{cases}. \end{aligned}$$

跳跃间断点 $x = 1$ 。

(11)

$$\begin{aligned} f(x) &= \lim_{t \rightarrow +\infty} \frac{x + e^{tx}}{1 + e^{tx}} \\ &= \begin{cases} x, & x < 0 \\ \frac{1}{2}, & x = 0. \\ 1, & x > 0 \end{cases} \end{aligned}$$

跳跃间断点。

5.

$$f(x) = \begin{cases} a + bx^2, & x \leq 0 \\ \frac{\sin(bx)}{x}, & x > 0 \end{cases}.$$

考虑 $x = 0$ 点,

$$\lim_{x \rightarrow 0^-} f(x) = a, \quad \lim_{x \rightarrow 0^+} f(x) = b.$$

由 $f(x)$ 在 $(-\infty, +\infty)$ 连续,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) \Leftrightarrow a = b.$$

9.

(2) 证明

考虑区间 $(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2})$, $k \in \mathbb{N}^*$. 记 $f(x) = x - \tan x$, 则

$$f\left(k\pi - \frac{\pi}{4}\right) = k\pi - \frac{\pi}{4} + 1 > 0.$$

由

$$\lim_{x \rightarrow (k\pi + \frac{\pi}{2})^-} \tan x = +\infty,$$

$$\forall M > 0, \exists X \in \left[k\pi, k\pi + \frac{\pi}{2}\right),$$

使得当 $x > X$ 时, $\tan x > M$.

令 $M = x$, 则 $\exists X \in [k\pi, k\pi + \frac{\pi}{2})$, 使得当 $x > X$ 时, $x - \tan x < 0$. 不妨取点

$$x_0 = \frac{X + k\pi + \frac{\pi}{2}}{2} > X,$$

则 $f(x_0) < 0$.

由零点存在性定理, 区间 $(k\pi - \frac{\pi}{4}, x_0) \subseteq (k\pi, k\pi + \frac{\pi}{2})$ 上至少有一个实根. 因为 $k \in \mathbb{N}^*$ 有无穷个, 所以 $x - \tan x = 0$ 有无穷个实根.

(3)

$x - \varepsilon \sin x = 1$ ($0 < \varepsilon < 1$) 在 $(-\infty, +\infty)$ 有且仅有一实根.

证明

记 $f(x) = x - \varepsilon \sin x - 1$.

1) 存在性:

$$f(3) = 3 - \varepsilon \sin 3 - 1 > 2 - 1 > 0,$$

$$f(-3) = -3 - \varepsilon \sin(-3) - 1 < -4 + 1 < 0.$$

由零点存在性定理, 至少存在一个实根.

2) 唯一性: 假设有两个根 x_1, x_2 ,

$$x_1 - \varepsilon \sin x_1 = x_2 - \varepsilon \sin x_2,$$

$$x_1 - x_2 - \varepsilon(\sin x_1 - \sin x_2) = 0,$$

$$\Leftrightarrow \frac{\varepsilon|\sin x_1 - \sin x_2|}{|x_1 - x_2|} = 1.$$

而 $|\sin x_1 - \sin x_2| \leq |x_1 - x_2|$,

$$\frac{\varepsilon|\sin x_1 - \sin x_2|}{|x_1 - x_2|} < 1,$$

矛盾。故有且仅有一个零点。

11.

$f(x)$ 在 $[0, 1]$ 连续, $0 < f(x) < 1$ 。要证 $\exists \xi \in (0, 1)$, 使得 $f(\xi) = \xi$ 。

记 $\varphi(x) = f(x) - x$,

$$\varphi(0) = f(0) > 0, \quad \varphi(1) = f(1) - 1 < 0.$$

由零点存在性定理, $\exists \xi \in (0, 1)$, 使得

$$\varphi(\xi) = 0 \Leftrightarrow f(\xi) = \xi.$$

12.

(1)

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1 + \alpha x)^a - (1 + \beta x)^b}{x} \\ &= \lim_{x \rightarrow 0} \frac{\left[1 + a\alpha x + \frac{a(a-1)}{2}(\alpha x)^2 + \dots\right] - \left[1 + b\beta x + \frac{b(b-1)}{2}(\beta x)^2 + \dots\right]}{x} \\ &= \lim_{x \rightarrow 0} \frac{[1 + a\alpha x + o(x)] - [1 + b\beta x + o(x)]}{x} \\ &= a\alpha - b\beta. \end{aligned}$$

(2)

$$\begin{aligned} & \lim_{x \rightarrow a} \frac{x^\alpha - a^\alpha}{x^\beta - a^\beta}, \quad (a > 0, \beta \neq 0) \\ &= \lim_{x \rightarrow a} \frac{a^\alpha}{a^\beta} \cdot \frac{\left(\frac{x}{a}\right)^\alpha - 1}{\left(\frac{x}{a}\right)^\beta - 1} \\ &= \lim_{x \rightarrow a} a^{\alpha-\beta} \cdot \frac{\alpha\left(\frac{x}{a} - 1\right)}{\beta\left(\frac{x}{a} - 1\right)} \\ &= \frac{\alpha}{\beta} a^{\alpha-\beta}. \end{aligned}$$

(5)

$$\begin{aligned} & \lim_{x \rightarrow +\infty} \frac{\ln(1 + 3^x)}{\ln(1 + 2^x)} \\ &= \lim_{x \rightarrow +\infty} \frac{\ln(3^x) + \ln\left(1 + \frac{1}{3^x}\right)}{\ln(2^x) + \ln\left(1 + \frac{1}{2^x}\right)} \\ &= \lim_{x \rightarrow +\infty} \frac{x \ln 3}{x \ln 2} = \frac{\ln 3}{\ln 2}. \end{aligned}$$

(8)

$$\begin{aligned}& \lim_{x \rightarrow +\infty} [\sin \ln(x+1) - \sin \ln x] \\&= \lim_{x \rightarrow +\infty} 2 \cos\left(\frac{\ln(x+1) + \ln x}{2}\right) \sin\left(\frac{\ln(x+1) - \ln x}{2}\right) \\&= \lim_{x \rightarrow +\infty} 2 \cos\left(\frac{1}{2} \ln[x(x+1)]\right) \sin\left(\frac{1}{2} \ln\left(\frac{x+1}{x}\right)\right).\end{aligned}$$

$2 \cos\left(\frac{1}{2} \ln[x(x+1)]\right)$ 为有界量, 而

$$\lim_{x \rightarrow +\infty} \sin\left(\frac{1}{2} \ln\left(\frac{x+1}{x}\right)\right) = 0.$$

有界量乘无穷小量仍无穷小, 故原极限为 0。

Chapter 1 Comprehensive Exercises Answer

2.

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{n^a}{n^b - (n-1)^b} &= 1995. \\
\Leftrightarrow \lim_{n \rightarrow \infty} n^{a-b} \cdot \frac{1}{1 - \left(1 - \frac{1}{n}\right)^b} \\
\Leftrightarrow \lim_{n \rightarrow \infty} n^{a-b} \cdot \frac{1}{1 - \left(1 - \frac{b}{n}\right)} \\
\Leftrightarrow \lim_{n \rightarrow \infty} n^{a-b+1} \cdot \frac{1}{b}.
\end{aligned}$$

要使极限收敛, 只有

$$\begin{cases} a - b + 1 = 0 \\ \frac{1}{b} = 1995 \end{cases} \Rightarrow \begin{cases} a = -\frac{1994}{1995} \\ b = \frac{1}{1995} \end{cases}.$$

3.

(1)

$$\lim_{x \rightarrow 0} \frac{3 \sin x + x^2 \cos\left(\frac{1}{x}\right)}{(1 + \cos x) \ln(1 + x)}.$$

 $\cos\left(\frac{1}{x}\right)$ 是有界量, x^2 是无穷小量。

$$\begin{aligned}
&\lim_{x \rightarrow 0} \frac{3 \sin x}{(1 + \cos x) \ln(1 + x)} \\
&= \lim_{x \rightarrow 0} \frac{3x}{(1 + \cos x)x} = \frac{3}{2}.
\end{aligned}$$

(2)

$$\begin{aligned}
&\lim_{x \rightarrow 0} \left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n} \right)^{\frac{1}{x}} \\
&= \lim_{x \rightarrow 0} e^{\frac{1}{x} \ln \left[\frac{a_1^x + a_2^x + \cdots + a_n^x}{n} \right]} \\
&= \lim_{x \rightarrow 0} e^{\frac{1}{x} \ln \left[\frac{a_1^{x-1} + a_2^{x-1} + \cdots + a_n^{x-1}}{n} + 1 \right]} \\
&= e^{\lim_{x \rightarrow 0} \frac{1}{x} \left[\frac{a_1^{x-1} + a_2^{x-1} + \cdots + a_n^{x-1}}{n} \right]} \\
&= e^{\lim_{x \rightarrow 0} \frac{1}{nx} (x \ln a_1 + x \ln a_2 + \cdots + x \ln a_n)} \\
&= e^{\frac{1}{n} (\ln a_1 + \ln a_2 + \cdots + \ln a_n)} \\
&= (a_1 a_2 \cdots a_n)^{\frac{1}{n}}.
\end{aligned}$$

4.

$$\begin{aligned}
& \lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 1} - ax + b \right) = 3. \\
&= \lim_{x \rightarrow \infty} \left(x - \frac{x-1}{x+1} - ax + b \right) \\
&= \lim_{x \rightarrow \infty} \left[(1-a)x + (b-1) + \frac{2}{x+1} \right] \\
&= \lim_{x \rightarrow \infty} [(1-a)x + (b-1)].
\end{aligned}$$

由极限收敛且为 3,

$$\begin{cases} 1-a=0 \\ b-1=3 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=4 \end{cases}.$$

11.

$f(x)$ 在 $[a, b]$ 连续, $a < c < d < b$, $k = f(c) + f(d)$ 。

(1) 证明

要证 $\exists \xi \in (a, b)$, 使得 $f(c) + f(d) = 2f(\xi)$,

$$\Leftrightarrow f(\xi) = \frac{f(c) + f(d)}{2}.$$

因为

$$\min\{f(c), f(d)\} < \frac{f(c) + f(d)}{2} < \max\{f(c), f(d)\},$$

所以由介值定理, $\exists \xi \in [c, d]$, 使得

$$f(\xi) = \frac{f(c) + f(d)}{2}.$$

(2) 证明

要证 $\exists \xi \in (a, b)$, 使得 $mf(c) + nf(d) = (m+n)f(\xi)$ 。由 (1), 相当于只要证明

$$s = \min\{f(c), f(d)\} < \frac{mf(c) + nf(d)}{m+n} < \max\{f(c), f(d)\} = t.$$

若 $s > \frac{m}{m+n}f(c) + \frac{n}{m+n}f(d)$, 不妨 $f(c) = s \leq f(d) = t$,

$$\Leftrightarrow s \geq \frac{ms + nt}{m+n} \Leftrightarrow s > t,$$

矛盾。

若 $t < \frac{m}{m+n}f(c) + \frac{n}{m+n}f(d)$,

$$\Leftrightarrow t < s,$$

矛盾。综上, 原命题成立。

16.

$$\begin{aligned}
\lim_{x \rightarrow +\infty} \frac{\arcsin(\sqrt{x^2 + \sqrt{x}} - x)}{\frac{C}{x^k}} &= 1. \\
&= \lim_{x \rightarrow +\infty} \frac{(\sqrt{x^2 + \sqrt{x}} - x)x^k}{C} \\
&= \lim_{x \rightarrow +\infty} \frac{\sqrt{x^{2k+2} + \sqrt{x^{4k+1}}} - x^{k+1}}{C}.
\end{aligned}$$

极限收敛, 则 $k = -1$,

$$\lim_{x \rightarrow +\infty} \frac{-1}{C} = 1.$$

所以 $k = C = -1$ 。

17.

$$\begin{aligned}
f(x) &= \frac{\sqrt{1 + \sin x + \sin^2 x} - (\alpha + \beta \sin x)}{\sin^2 x}. \\
\lim_{x \rightarrow 0} \frac{\sqrt{1 + \sin x + \sin^2 x} - (\alpha + \beta \sin x)}{\sin^2 x} \\
&= \lim_{x \rightarrow 0} \frac{1 + \sin x + \sin^2 x - \alpha^2 - \beta^2 \sin x - 2\alpha\beta \sin x}{\sin^2 x [\sqrt{1 + \sin x + \sin^2 x} + (\alpha + \beta \sin x)]}. \\
&= \lim_{x \rightarrow 0} \frac{(1 - \alpha^2) + (1 - 2\alpha\beta) \sin x + (1 - \beta^2) \sin^2 x}{\sin^2 x [\sqrt{1 + \sin x + \sin^2 x} + (\alpha + \beta \sin x)]}. \\
&= \lim_{x \rightarrow 0} \frac{(1 - \alpha^2) + (1 - 2\alpha\beta) \sin x + (1 - \beta^2) \sin^2 x}{\sin^2 x (1 + \alpha)}.
\end{aligned}$$

由极限存在,

$$\begin{cases} 1 - \alpha^2 = 0 \\ 1 - 2\alpha\beta = 0 \\ 1 + \alpha \neq 0 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = \frac{1}{2} \end{cases}.$$

18.

$$\lim_{x \rightarrow 0} \frac{\sqrt{1 + \frac{1}{x}f(x)} - 1}{cx^2} = 1.$$

即

$$\sqrt{1 + \frac{1}{x}f(x)} \sim cx^2 + 1,$$

$$1 + \frac{1}{x}f(x) \sim (cx^2 + 1)^2 = c^2x^4 + 1 + 2cx^2,$$

$$f(x) \sim cx^5 + 2cx^3 \sim \tau x^k.$$

则 $\tau = c$, $k = 5$ 。