

$$28 \quad (2) \quad A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\text{by def. } A^{-1} = \frac{A^*}{|A|}$$

$$A_{11} = (-1)^2 \cdot 1 \cdot 0 = 0$$

$$|A| = (-1) \cdot (-1)^{1+3} \cdot (-2-1) = 3$$

$$A_{21} = (-1)^3 \cdot 2 \cdot (-1) = 2$$

$$A_{31} = (-1)^4 \cdot 1 \cdot 1 = 1$$

$$A_{12} = (-1)^3 \cdot 1 \cdot 0 = 0$$

$$\Rightarrow A^* = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & 2 \\ 3 & 0 & 0 \end{bmatrix}$$

$$A_{22} = (-1)^4 \cdot 1 \cdot 1 = 1$$

$$A_{32} = (-1)^3 \cdot (-1) \cdot 2 = 2$$

$$A_{13} = (-1)^4 \cdot (-1) \cdot (-3) = 3$$

$$A_{23} = 0$$

$$A_{33} = 0$$

$$A^{-1} = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 0 & 1/3 & 2/3 \\ 1 & 0 & 0 \end{bmatrix}$$

$$29 \quad \text{Find } X, \quad \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} X \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 4 \\ -1 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1/3 & -2/3 \\ 1/6 & 1/6 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \quad B^{-1} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$A^{-1} A X B B^{-1} = A^{-1} \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} B^{-1}$$

$$X = \begin{bmatrix} 1/3 & -2/3 \\ 1/6 & 1/6 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/2 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/4 & 0 \end{bmatrix}$$

$$32 \quad A \in \mathbb{P}^{n \times n}, \exists k \in \mathbb{N}^*, k \geq 2 \quad A^k = 0$$

pf.

$$\begin{aligned} & (E - A)(E + A + A^2 + \dots + A^{k-1}) \\ &= E + A + A^2 + \dots + A^{k-1} - (AE + A^2 + A^3 + \dots + A^k) \\ &= E + A - A + A^2 - A^2 + \dots + A^{k-1} - A^{k-1} - A^k \\ &= E \end{aligned}$$

by def, $(E - A)$ is invertible, and $(E - A)^{-1} = E + A + \dots + A^{k-1}$

$$34 \quad A \in \mathbb{P}^{n \times n}, \quad A^2 + A - 4E = 0$$

$$\text{pf } 1) \because A^2 + A - 4E = 0$$

$$\therefore A(A + E) = 4E$$

$$\Leftrightarrow A \cdot \left(\frac{A + E}{4} \right) = E$$

by def, A is invertible, $A^{-1} = \frac{A + E}{4}$

$$2) \because A^2 + A - 4E = 0$$

$$\therefore (A - E)(A + 2E)$$

$$= A^2 + (2-1)A - 2E$$

$$\text{for } k-1 = 1 \Rightarrow k = 2.$$

$$A^2 + (2-1)A - 2E = 2E$$

$$\Leftrightarrow (A - E)(A + 2E) = 2E$$

$$(A - E) \cdot \left(\frac{A + 2E}{2} \right) = E$$

by def, $(A - E)$ is invertible. $(A - E)^{-1} = \frac{A + 2E}{2}$

37. $A \in \mathbb{R}^{n \times n}$ prove that $(A^*)^* = |A|^{-n} A$

pf. $A^* A = A A^* = |A| E$

$$|A^*| = |A|^{n-1}$$

1) if A is not invertible $\Leftrightarrow |A| = 0 \Leftrightarrow r(A) < n$

$$\Leftrightarrow (A^*)^* = 0$$

① $r(A) < n-1$, $r(A^*) = 0 \Rightarrow A^* = 0$, $(A^*)^* = 0$

② $r(A) = n-1 > 1$, $r(A^*) = 1 \Rightarrow r((A^*)^*) = 0$, $(A^*)^* = 0$

2) if A is invertible. $\Leftrightarrow |A| \neq 0$

$$A^* = |A| A^{-1}$$

$$(A^*)^* = (|A| A^{-1})^* = |A|^{n-1} (A^{-1})^*$$

$$(A^{-1})^* = |A^{-1}| \cdot A = \frac{A}{|A|}$$

$$(A^*)^* = |A|^{n-2} A$$

38. $|A| = 3$, $n = 3$

① $|A^{-1}| = \frac{1}{|A|} = \frac{1}{3}$

② $|A^*| = |A|^{n-1} = 3^2 = 9$

③ $|-2A| = (-2)^3 \cdot |A| = -24$

④ $|(-3A)^{-1}| = \frac{1}{|-3A|} = \frac{1}{(-3)^3 |A|} = -\frac{1}{81}$

⑤ $|\frac{1}{3} A^* - 4 A^{-1}|$

$$= |\frac{1}{3} A^* - \frac{4A^*}{3}|$$

$$= |-A^*|$$

$$= (-1)^3 |A^*|$$

$$= -9$$

$$\begin{aligned} \textcircled{6} (A^*)^{-1} &= \frac{(A^*)^*}{|A^*|} \\ &= \frac{|A|^{n-2} A}{|A^*|} \\ &= \frac{1}{3} A \end{aligned}$$

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$$A = \left[\begin{array}{ccc|cc} 1 & -2 & 7 & 0 & 0 \\ -1 & 3 & 6 & 0 & 0 \\ -3 & 2 & -5 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right] = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}$$

$$B = \left[\begin{array}{ccc|cc} 3 & 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & 3 & 4 \\ 0 & 0 & 3 & 5 & 6 \\ \hline 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 5 & 1 \end{array} \right] = \begin{bmatrix} 3E_3 & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} 3E_3 & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 3A_{11}E_3 & A_{11}B_{12} \\ 0 & A_{22}B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -6 & 21 & 30 & 36 \\ -3 & 9 & 18 & 26 & 46 \\ -9 & 6 & -15 & -22 & -28 \\ 0 & 0 & 0 & 13 & 6 \\ 0 & 0 & 0 & 25 & 5 \end{bmatrix}$$